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MATHEMATICS
TARGET IIT JEE

CONIC SECTION

(PARABOLA, ELLIPSE, HYPERBOLA)

THEORY AND EXERCISE BOOKLET

CONTENTS

PARABOLA

ELLIPSE

HYPERBOLA

ANSWER KEY

S.NO.	TOPIC	PAGE NO.

JEE Syllabus :

Equations of a parabola, ellipse and hyperbola in standard form, their foci, directrixes and eccentricity, parametric equations, equations of tangent and normal.

PARABOLA

A. CONIC SECTION

A conic section, or conic is the locus of a point which moves in a plane so that its distance from a fixed point is in a constant ratio to its perpendicular distance from a fixed straight line.

(a) The fixed point is called the FOCUS.

(b) The fixed straight line is called the DIRECTRIX.

(c) The constant ratio is called the ECCENTRICITY denoted by 'e'.

(d) The line passing through the focus & perpendicular to the directrix is called the AXIS.

(e) A point of intersection of a conic with its axis is called a VERTEX.

B. GENERAL EQUATION OF A CONIC : FOCAL DIRECTRIX PROPERTY

The general equation of a conic with focus (p, q) & directrix $lx + my + n = 0$ is

$(l^2 + m^2)\,[(x - p)^2 + (y - q)^2] = e^2\,(lx + my + n)^2 \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

C. DISTINGUISHING BETWEEN THE CONIC

The nature of the conic section depends upon the position of the focus S w.r.t. the directrix & also upon the value of the eccentricity e. Two different cases arise.

Case (i) When the focus lies on the directrix

In this case $D \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ & the general equation of a conic represents a pair of straight lines and if :

$e > 1$ the lines will be real & distinct intersecting at S.

$e = 1$ the lines will be coincident.

$e < 1$ the lines will be imaginary.

Case (ii) When the focus does not lie on the directrix

The conic represents :

a parabola	**an ellipse**	**a hyperbola**	**a rectangular hyperbola**
$e = 1$; $D \neq 0$	$0 < e < 1$; $D \neq 0$	$D \neq 0$; $e > 1$;	$e > 1$; $D \neq 0$
$h^2 = ab$	$h^2 < ab$	$h^2 > ab$	$h^2 > ab$; $a + b = 0$

D. PARABOLA

A parabola is the locus of a point which moves in a plane, such that its distance from a fixed point (focus) is equal to its perpendicular distance from a fixed straight line (directrix).

Standard equation of a parabola is $y^2 = 4ax$. For this parabola :

(i) Vertex is (0, 0) **(ii)** Focus is (a, 0) **(iii)** Axis is $y = 0$ **(iv)** Directrix is $x + a = 0$

(a) Focal distance : The distance of a point on the parabola from the focus is called the FOCAL DISTANCE OF THE POINT.

(b) Focal chord : A chord of the parabola, which passes through the focus is called a FOCAL CHORD.

(c) Double ordinate : A chord of the parabola perpendicular to the axis of the symmetry is called a DOUBLE ORDINATE.

(d) Latus rectum : A double ordinate passing through the focus or a focal chord perpendicular to the axis of parabola is called the LATUS RECTUM. For $y^2 = 4ax$.

- Length of the latus rectum = 4a.
- Length of the semi latus rectum = 2a.
- Ends of the latus rectum are L (a, 2a) & L (a, –2a)

Note that :

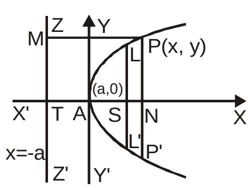
(i) Perpendicular distance from focus on directrix = half the latus rectum.

(ii) Vertex is middle point of the focus & the point of intersection of directrix & axis.

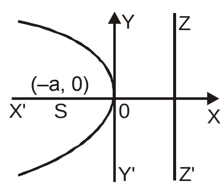
(iii) Two parabolas are said to be equal if they have the same latus rectum.

E. TYPE OF PARABOLA

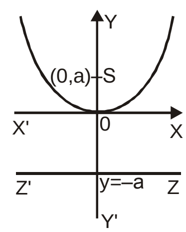
Four standard forms of the parabola are $y^2 = 4ax$; $y^2 = -4ax$; $x^2 = 4ay$; $x^2 = -4ay$



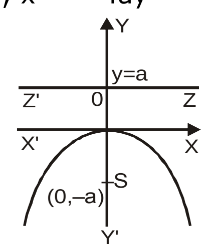
$y^2 = 4ax$



$y^2 = -4ax$



$x^2 = 4ay$



$x^2 = -4ay$

Parabola	Vertex	Focus	Axis	Directrix	Length of Latus rectum	Ends of Latus rectum	Parametric eqution	Focal length
$y^2 = 4ax$	(0, 0)	(a, 0)	y = 0	x = –a	4a	(a, ±2a)	$(at^2, 2at)$	x + a
$y^2 = -4ax$	(0, 0)	(–a, 0)	y = 0	x = a	4a	(–a, ±2a)	$(-at^2, 2at)$	x – a
$x^2 = +4ay$	(0, 0)	(0, a)	x = 0	y = –a	4a	(± 2a, a)	$(2at, at^2)$	y + a
$x^2 = -4ay$	(0, 0)	(0, –a)	x = 0	y = a	4a	(± 2a, –a)	$(2at, -at^2)$	y – a
$(y-k)^2 = 4a(x-h)$	(h, k)	(h + a, k)	y = k	k + a – h = 0	4a	(h + a, k ± 2a)	$(h + at^2, k + 2at)$	x – h + a
$(x-p)^2 = 4b(y-q)$	(p, q)	(p, b + q)	x = p	y + b – q = 0	4b	(p ± 2a, q + a)	$(p + 2at, q + at^2)$	y – q + b

F. PARAMETRIC REPRESENTATION

The simplest & the best form of representing the co-ordinates of a point on the parabola is $(at^2, 2at)$. The equation $x = at^2$ & $y = 2at$ together represents the parabola $y^2 = 4ax$, t being the parameter.

Ex.1 Find the vertex, axis, directrix, focus, latus rectum and the tangent at vertex for the parabola $9y^2 - 16x - 12y - 57 = 0$.

Sol. The given equation can be rewritten as $\left(y-\frac{2}{3}\right)^2 = \frac{16}{9}\left(x+\frac{61}{16}\right)$ which is of the form $Y^2 = 4AX$.

Hence the vertex is $\left(-\frac{61}{16}, \frac{2}{3}\right)$. The axis the $y - \frac{2}{3} = 0 \Rightarrow y = \frac{2}{3}$

The directrix is $X + A = 0 \Rightarrow x + \frac{61}{16} + \frac{4}{9} = 0 \Rightarrow x = -\frac{613}{144}$

The focus is $X = A$ and $Y = 0 \Rightarrow x + \frac{61}{16} = \frac{4}{9}$ and $y - \frac{2}{3} = 0 \Rightarrow \left(-\frac{485}{144}, \frac{2}{3}\right)$ is the focus

Length of the latus rectum $= 4A = \frac{16}{9}$. The tangent at the vertex is $X = 0 \Rightarrow x = -\frac{61}{16}$.

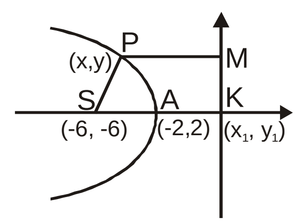
Ex.2 The length of latus rectum of a parabola, whose focus is (2, 3) and directrix is the line $x - 4y + 3 = 0$ is

Sol. The length of latus rectum = 2 × perp. from focus to the directrix $= 2 \times \left|\frac{2-4(3)+3}{\sqrt{(1)^2+(4)^2}}\right| = \frac{14}{\sqrt{17}}$

Ex.3 Find the equation of the parabola whose focus is (–6, –6) and vertex (–2, 2).

Sol. Let S(–6, –6) be the focus and A(–2, 2) is vertex of the parabola. On SA take a point $K(x_1, y_1)$ such that SA = AK. Draw KM perpendicular on SK. Then KM is the directrix of the parabola.

Since A bisects SK, $\left(\frac{-6+x_1}{2}, \frac{-6+y_1}{2}\right) = (-2, 2)$



$\Rightarrow -6 + x_1 = -4$ and $-6 + y_1 = 4$ or $(x_1, y_1) = (2, 10)$

Hence the equation of the directrix KM is

$y - 10 = m(x + 2)$ ....(i)

Also gradient of SK $= \frac{10-(-6)}{2-(-6)} = \frac{16}{8} = 2; \Rightarrow m = \frac{-1}{2}$

$y - 10 = \frac{-1}{2}(x - 2)$ (from (i)) $\Rightarrow x + 2y - 22 = 0$ is the directrix

Next, let PM be a perpendicular on the directrix KM from any point P(x, y) on the parabola. From

SP = PM, the equation of the parabola is $\sqrt{\{(x+6)^2+(y+6)^2\}} = \frac{|x+2y-22|}{\sqrt{(1^2+2^2)}}$

or $5(x^2 + y^2 + 12x + 12y + 72) = (x + 2y - 22)^2$

or $4x^2 + y^2 - 4xy + 104x + 148y - 124 = 0$ or $(2x - y)^2 + 104x + 148y - 124 = 0$.

Ex.4 The extreme points of the latus rectum of a parabola are (7, 5) and (7, 3). Find the equation of the parabola.

Sol. Focus of the parabola is the mid-point of the latus rectum.

$\Rightarrow$ S is (7, 4). Also axis of the parabola is perpendicular to the latus rectum and passes through the focus. Its equation is $y - 4 = \frac{0}{5-3}(x - 7) \Rightarrow y = 4$

Length of the latus rectum = (5 – 3) = 2

Hence the vertex of the parabola is at a distance 2/4 = 0.5 from the focus. We have two parabolas, one concave rightward and the other concave leftward.

The vertex of the first parabola is (6.5, 4) and its equation is $(y - 4)^2 = 2(x - 6.5)$ and it meets the x-axis at (14.5, 0). The equation of the second parabola is $(y - 4)^2 = -2(x - 7.5)$. It meets the x-axis at (–0.5, 0).

G. POSITION OF A POINT RELATIVE TO A PARABOLA

The point (x_1, y_1) lies outside, on or inside the parabola $y^2 = 4ax$ according as the expression $y_1^2 - 4ax_1$ is positive, zero or negative.

Ex.5 Find the value of α for which the point $(\alpha - 1, \alpha)$ lies inside the parabola $y^2 = 4x$.

Sol. $\because$ Point $(\alpha - 1, \alpha)$ lies inside the parabola $y^2 = 4x$

$\therefore \quad y_1^2 - 4x_1 < 0 \quad \Rightarrow \quad \alpha^2 - 4(\alpha - 1) < 0 \quad \Rightarrow \quad \alpha^2 - 4\alpha + 4 < 0 \Rightarrow \quad (\alpha - 2)^2 < 0 \Rightarrow \alpha \in \phi$

H. CHORD JOINING TWO POINTS

The equation of a chord of the parabola $y^2 = 4ax$ joining its two points $P(t_1)$ and $Q(t_2)$ is $y(t_1 + t_2) = 2x + 2at_1t_2$

Note :

(i) If PQ is focal chord then $t_1t_2 = -1$.

(ii) Extremities of focal chord can be taken as $(at^2, 2at)$ & $\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$

Ex.6 Through the vertex O of a parabola $y^2 = 4x$ chords OP and OQ are drawn at right angles to one another. Show that for all position of P, PQ cuts the axis of the parabola at a fixed point.

Sol. The given parabola is $y^2 = 4x$ ....(i)

Let $P \equiv (t_1^2, 2t_1)$, $Q \equiv (t_2^2, 2t_2)$. Slope of OP = $\frac{2t_1}{t_1^2} = \frac{2}{t_1}$ and slope of OQ = $\frac{2}{t_2}$

Since OP $\perp$ OQ, $\frac{4}{t_1t_2} = -1$ or $t_1t_2 = -4$ ....(ii)

The equation of PQ is $y(t_1 + t_2) = 2(x + t_1t_2)$

$\Rightarrow \quad y\left(t_1 - \frac{4}{t_1}\right) = 2(x - 4) \qquad$ [from (ii)] $\qquad \Rightarrow \quad 2(x - 4) - y\left(t_1 - \frac{4}{t_1}\right) = 0 \qquad \Rightarrow \quad L_1 + \lambda L_2 = 0$

$\therefore$ variable line PQ passes through a fixed point which is point of intersection of $L_1 = 0$ & $L_2 = 0$ i.e. (4, 0)

I. LINE & A PARABOLA

(a) The line y = mx + c meets the parabola $y^2 = 4ax$ in two points real, coincident or imaginary according as a > = < cm $\Rightarrow$ condition of tangency is, $c = \frac{a}{m}$.

Note : Line y = mx + c will be tangent to parabola $x^2 = 4ay$ if $c = -am^2$

(b) Length of the chord intercepted by the parabola $y^2 = 4ax$ on the line y = mx + c is

$$\left(\frac{4}{m^2}\right)\sqrt{a(1+m^2)(a-mc)}.$$

Note : length of the focal chord making an angle α with the x-axis is $4a\,\text{cosec}^2\,\alpha$.

Ex.7 If the line $y = 3x + \lambda$ intersect the parabola $y^2 = 4x$ at two distinct points then set of values of λ is

Sol. Putting value of y from the line in the parabola –

$(3x + \lambda)^2 = 4x \Rightarrow 9x^2 + (6\lambda - 4)x + \lambda^2 = 0$ $\quad\because$ line cuts the parabola at two distinct points

$\therefore D > 0 \Rightarrow 4(3\lambda - 2)^2 - 4.9\lambda^2 > 0 \Rightarrow 9\lambda^2 - 12\lambda + 4 - 9\lambda^2 > 0 \Rightarrow \lambda < 1/3$ Hence, $\lambda \in (-\infty, 1/3)$

J. TANGENT TO THE PARABOLA $y^2 = 4ax$

(a) Point form : Equation of tangent to the given parabola at its point (x_1, y_1) is $yy_1 = 2a(x + x_1)$

(b) Slope form : Equation of tangent to the given parabola whose slope is 'm', is

$$y = mx + \frac{a}{m},\ (m \neq 0) \text{ \& Point of contact is } \left(\frac{a}{m^2}, \frac{2a}{m}\right)$$

(c) Parametric form : Equation of tangent to the given parabola at its point P(t), is $ty = x + at^2$

Note : Point of intersection of the tangents at the point t_1 & t_2 is $[at_1 t_2, a(t_1 + t_2)]$, (i.e. G.M. and A.M. of abscissa and ordinates of the points)

Ex.8 A tangent to the parabola $y^2 = 8x$ makes an angle of 45º with the straight line y = 3x + 5. Find its equation and its point of contact.

Sol. Let the slope of the tangent be m

$$\therefore \tan 45^\circ = \left|\frac{3-m}{1+3m}\right| \Rightarrow 1 + 3m = \pm(3 - m) \quad \therefore \quad m = -2 \text{ or } \frac{1}{2}$$

As we know that equation of tangent of slope m to the parabola $y^2 = 4ax$ is $y = mx + \frac{a}{m}$ and point of contact is $\left(\frac{a}{m^2}, -\frac{2a}{m}\right)$

for m = –2, equation of tangent is y = –2x – 1 and point of contact is $\left(\frac{1}{2}, -2\right)$

for $m = \frac{1}{2}$, equation of tangent is $y = \frac{1}{2}x + 4$ and point of contact is (8, 8)

Ex.9 Find the equation to the tangents to the parabola $y^2 = 9x$ which go through the point (4, 10).

Sol. Equation of tangent to parabola $y^2 = 9x$ is $y = mx + \frac{9}{4m}$

Since it passes through (4, 10) $\therefore 10 = 4m + \frac{9}{4m} \Rightarrow 16m^2 - 40m + 9 = 0 \Rightarrow m = \frac{1}{4}, \frac{9}{4}$

$\therefore$ equation of tangent's are $y = \frac{x}{4} + 9$ & $y = \frac{9}{4}x + 1$

Ex.10 Find the locus of the point P from which tangents are drawn to the parabola $y^2 = 4ax$ having slopes m_1 and m_2 such that

(i) $m_1^2 + m_2^2 = \lambda$ (constant) **(ii)** $\theta_1 - \theta_2 = \theta_0$ (constant)

where θ_1 and θ_2 are the inclinations of the tangents from positive x-axis.

Sol. Equation of tangent to $y^2 = 4ax$ is $y = mx + a/m$

Let it passes through P(h, k). $\therefore$ $m^2h - mk + a = 0$

(i) $m_1^2 + m_2^2 = \lambda \Rightarrow (m_1 + m_2)^2 - 2m_1m_2 = \lambda \Rightarrow \frac{k^2}{h^2} - 2.\frac{a}{h} = \lambda$

$\therefore$ locus of P(h, k) is $y^2 - 2ax = \lambda x^2$

(ii) $\theta_1 - \theta_2 = \theta_0$

$\tan(\theta_1 - \theta_2) = \tan\theta_0 \Rightarrow \frac{m_1 - m_2}{1 + m_1m_2} = \tan\theta_0 \Rightarrow (m_1 + m_2)^2 - 4m_1m_2 = \tan^2\theta_0\,(1 + m_1m_2)^2$

$\frac{k^2}{h^2} - \frac{4a}{h} = \tan^2\theta_0\left(1 + \frac{a}{h}\right)^2 \Rightarrow k^2 - 4ah = (h + a)^2 \tan^2\theta_0$

$\therefore$ locus of P(h, k) is $y^2 - 4ax = (x + a)^2 \tan^2\theta_0$

K. DIRECTOR CIRCLE

Locus of the point of intersection of the perpendicular tangents to the parabola $y^2 = 4ax$ is called the DIRECTOR CIRCLE. It's equation is $x + a = 0$ which is parabola's own directrix.

Ex.11 The angle between the tangents drawn from a point (–a, 2a) to $y^2 = 4ax$ is

Sol. The given point (–a, 2a) lies on the directrix $x = -a$ of the parabola $y^2 = 4ax$. Thus, the tangents are at right angle.

Ex.12 The circle drawn with variable chord $x + ay - 5 = 0$ (a being a parameter) of the parabola $y^2 = 20x$ as diameter will always touch the line

Sol. Clearly $x + ay - 5 = 0$ will always pass through the focus of $y^2 = 20x$ i.e. (5, 0). Thus the drawn circle will always touch the directrix of the parabola i.e.. the line $x + 5 = 0$.

L. NORMAL TO THE PARABOLA $y^2 = 4ax$

(a) Point form : Equation of normal to the given parabola at its point (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

(b) Slope form : Equation of normal to the given parabola whose slope is 'm', is

$y = mx - 2am - am^3$ & foot of the normal is $(am^2, -2am)$

(c) Parametric form : Equation of normal to the given parabola at its point P(t), is

$y + tx = 2at + at^3$

Note :

(i) Point of intersection of normals at t_1 & t_2 is, $(a(t_1^2 + t_2^2 + t_1t_2 + 2), -at_1t_2(t_1 + t_2))$.

(ii) If the normal to the parabola $y^2 = 4ax$ at the point t_1, meets the parabola again at the point t_2, then $t_2 = -\left(t_1 + \frac{2}{t_1}\right)$.

(iii) If the normals to the parabola $y^2 = 4ax$ at the points t_1 & t_2 intersect again on the parabola at the point 't_3' then $t_1t_2 = 2$; $t_3 = -(t_1 + t_2)$ and the line joining t_1 & t_2 passes through a fixed point $(-2a, 0)$.

(iv) If normal drawn to a parabola passes through a point P(h, k) then $k = mh - 2am - am^3$ i.e. $am^3 + m(2a - h) + k = 0$.

This gives $m_1 + m_2 + m_3 = 0$; $m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a-h}{a}$; $m_1m_2m_3 = \frac{-k}{a}$

where m_1, m_2, & m_3 are the slopes of the three concurrent normals :

- Algebraic sum of slopes of the three concurrent normals is zero
- Algebraic sum of ordinates of the three co-normal points on the parabola is zero.
- Centroid of the Δ formed by three co-normal points lies on the axis of parabola (x-axis)

Ex.13 Prove that the normal chord to a parabola $y^2 = 4ax$ at the point whose ordinate is equal to abscissa subtends a right angle at the focus.

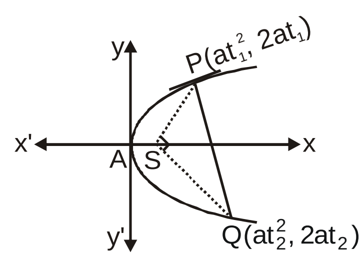
Sol. Let the normal at $P\left(at_1^2, 2at_1\right)$ meet the curve at $Q\left(at_2^2, 2at_2\right)$

$\therefore$ PQ is a normal chord and $t_2 = -t_1 - \frac{2}{t_1}$ ....(i)

By given condition $2at_1 = at_1^2$

$\therefore$ $t_1 = 2$ from equation (i), $t_2 = -3$

then P(4a, 4a) and Q(9a, –6a) but focus S(a, 0)



$\therefore$ Slope of SP $= \frac{4a-0}{4a-a} = \frac{4a}{3a} = \frac{4}{3}$

and Slope of SQ $= \frac{-6a-0}{9a-a} = \frac{-6a}{8a} = -\frac{3}{4}$

$\therefore$ Slope of SP × Slope of SQ $= \frac{4}{3} \times -\frac{3}{4} = -1$ $\therefore$ $\angle PSQ = \pi/2$

i.e. PQ subtends a right angle at the focus S.

Ex.14 If two normals drawn from any point to the parabola $y^2 = 4ax$ make angle α and β with the axis such that $\tan \alpha \,.\, \tan \beta = 2$, then find the locus of this point,

Sol. Let the point is (h, k). The equation of any normal to the parabola $y^2 = 4ax$ is $y = mx - 2am - am^3$ passes through (h, k) $\Rightarrow$ $k = mh - 2am - am^3$ $\Rightarrow$ $am^3 + m(2a - h) + k = 0$ ....(i)

m_1, m_2, m_3 are roots of the equation, then $m_1 . m_2 . m_3 = -\frac{k}{a}$ but $m_1m_2 = 2$, $m_3 = -\frac{k}{2a}$

m_3 is root of (i) $\therefore$ $a\left(-\frac{k}{2a}\right)^3 - \frac{k}{2a}(2a - h) + k = 0$ $\Rightarrow$ $k^2 = 4ah$. Thus locus is $y^2 = 4ax$.

Ex.15 Three normals are drawn from the point (14, 7) to the curve $y^2 - 16x - 8y = 0$. Find the coordinates of the feet of the normals.

Sol. The given parabola is $y^2 - 16x - 8y = 0$ ....(i)

Let the co-ordinates of the feet of the normal from (14, 7) be $P(\alpha, \beta)$. Now the equation of the tangent at $P(\alpha, \beta)$ to parabola (i) is

$y\beta - 8(x + \alpha) - 4(y + \beta) = 0$ or $(\beta - 4)y = 8x + 8a + 4\beta$ ....(ii)

Its slope $= \frac{8}{\beta - 4}$

Equation of the normal to parabola (i) at (α, β) is $y - \beta = \frac{4 - \beta}{8}(x - \alpha)$

It passes through (14, 7) $\Rightarrow$ $7 - \beta = \frac{4 - \beta}{8}(14 - \alpha)$ $\Rightarrow$ $\alpha = \frac{6\beta}{\beta - 4}$ ....(iii)

Also (α, β) lies on parabola (i) i.e. $\beta^2 - 16\alpha - 8\beta = 0$ ....(iv)

Putting the value of α from (iii) in (iv), we get $\beta^2 - \frac{96\beta}{\beta - 4} - 8\beta = 0$

$\Rightarrow$ $\beta^2(\beta - 4) - 96\beta - 8\beta(\beta - 4) = 0$ $\Rightarrow$ $\beta(\beta^2 - 4\beta - 96 - 8\beta + 32) = 0$

$\Rightarrow$ $\beta(\beta^2 - 12\beta - 64) = 0$ $\Rightarrow$ $\beta(\beta - 16)(\beta + 4) = 0$ $\Rightarrow$ $\beta = 0, 16, -4$

from (iii), $\alpha = 0$ when $\beta = 0$; $\alpha = 8$, when $\beta = 16$; $\alpha = 3$ when $\beta = -4$

Hence the feet of the normals are (0, 0) (8, 16) and (3, –4)

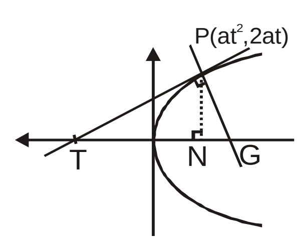
M. LENGTH OF SUBTANGENT & SUBNORMAL

PT and PG are the tangent and normal respectively at the point P to the parabola $y^2 = 4ax$. Then

TN = length of subtangent = twice the abscisse of the point P

(Subtangent is always bisected by the vertex)

NG = length of subnormal which is constant for all points on the parabola & equal to its semi latusrectum (2a).



N. PAIR OF TANGENTS

The equation of the pair of tangents which can be drawn from any point $P(x_1, y_1)$ out side the parabola to the parabola $y^2 = 4ax$ is given by : $SS_1 = T^2$ where :
$S \equiv y^2 - 4ax$; $S_1 \equiv y_1^2 - 4ax_1$; $T \equiv yy_1 - 2a(x + x_1)$.

O. CHORD OF CONTACT

Equation of the chord of contact of tangents drawn from a point $P(x_1, y_1)$ is $yy_1 = 2a(x + x_1)$
Remember that the area of the triangle formed by the tangents from the point (x_1, y_1) & the chord of contact is $\frac{(y_1^2 - 4ax_1)^{3/2}}{2a}$. Also note that the chord of contact exists only if the point P is not inside.

Ex.16 If the line $x - y - 1 = 0$ intersect the parabola $y^2 = 8x$ at P & Q, then find the point of intersection of tangents at P & Q.

Sol. Let (h, k) be point of intersection of tangents then chord of contact is
$yk = 4(x + h) \Rightarrow 4x - yk + 4h = 0$ ....(i)
But given line is $\Rightarrow x - y - 1 = 0$ ....(ii)
Comparing (i) and (ii), $\therefore \frac{4}{1} = \frac{-k}{-1} = \frac{4h}{-1} \Rightarrow h = -1, k = 4$ $\therefore$ point $\equiv (-1, 4)$

Ex.17 Find the locus of point whose chord of contact w.r.t. to the parabola $y^2 = 4bx$ is the tangent of the parabola $y^2 = 4ax$.

Sol. Equation of tangent to $y^2 = 4ax$ is $y = mx + \frac{a}{m}$ ....(i)
Let it is chord of contact for parabola $y^2 = 4bx$ w.r.t. the point P(h, k)
$\therefore$ Equation of chord of contact is $yk = 2b(x + h) \Rightarrow y = \frac{2b}{k}x + \frac{2bh}{k}$ ....(ii)
From (i) & (ii), $m = \frac{2b}{k}, \frac{a}{m} = \frac{2bh}{k} \Rightarrow a = \frac{4b^2h}{k^2}$ $\therefore$ locus of P is $y^2 = \frac{4b^2}{a}x$.

P. CHORD WITH A GIVEN MIDDLE POINT

Equation of the chord of the parabola $y^2 = 4ax$ whose middle point is (x_1, y_1) is $y - y_1 = \frac{2a}{y_1}(x - x_1)$.
The reduced to $T = S_1$ where $T \equiv yy_1 - 2a(x + x_1)$ & $S_1 \equiv y_1^2 - 4ax_1$.

Ex.18 Find the locus of middle of the chord of the parabola $y^2 = 4ax$ which pass through a given (p, q).

Sol. Let P(h, k) be the mid point of chord of the parabola $y^2 = 4ax$.
so equation of chord is $yk - 2a(x + h) = k^2 - 4ah$. Since it passes through (p, q)
$\therefore qk - 2a(p + h) = k^2 - 4ah$ $\therefore$ Required locus is $y^2 - 2ax - qy + 2ap = 0$.

Ex.19 Find the locus of the middle point of a chord of a parabola $y^2 = 4ax$ which subtends a right angle at the vertex.

Sol. The equation of the chord of the parabola whose middle point is (α, β) is

$$y\beta - 2a(x + \alpha) = \beta^2 - 4a\alpha \Rightarrow \quad y\beta - 2ax = \beta^2 - 2a\alpha \quad \text{or} \quad \frac{y\beta - 2ax}{\beta^2 - 2a\alpha} = 1 \quad \text{...........(i)}$$

Now, the equation of the pair of the lines OP and OQ joining the origin O i.e. the vertex to the points of intersection P and Q of the chord with the parabola $y^2 = 4ax$ is obtained by making the equation homogeneous by means of (i). Thus the equation of lines OP and OQ is $y^2 = \frac{4ax(y\beta - 2ax)}{\beta^2 - 2a\alpha}$

$$\Rightarrow \quad y^2(\beta^2 - 2a\alpha) - 4a\beta xy + 8a^2x^2 = 0$$

If the lines OP and OQ are at right angles, then the coefficient of x^2 + the coefficient of $y^2 = 0$ Therefore, $\beta^2 - 2a\alpha + 8a^2 = 0 \Rightarrow \beta^2 = 2a(\alpha - 4a)$. Hence the locus of (α, β) is $y^2 = 2a(x - 4a)$

Q. AN IMPORTANT CONCEPT

If a family of straight lines can be represented by an equation $\lambda^2P + \lambda Q + R = 0$ where λ is a parameter and P, Q, R are linear functions of x and y then the family of lines will be tangent to the curve $Q^2 = 4PR$.

Ex.20 If the equation $m^2(x + 1) + m(y - 2) + 1 = 0$ represents a family of lines, where 'm' is parameter then find the equation of the curve to which these lines will always be tangents.

Sol. $m^2(x + 1) + m(y - 2) + 1 = 0$. The equation of the curve to which above lines will always be tangents can be obtained by equating its discriminant to zero.

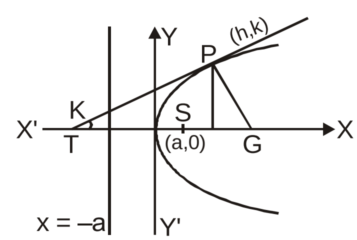
$$\therefore \quad (y - 2)^2 - 4(x + 1) = 0 \quad \Rightarrow \quad y^2 - 4y + 4 - 4x - 4 = 0 \quad \Rightarrow \quad y^2 = 4(x + y)$$

R. DIAMETER

The locus of the middle points of a system of parallel chords of a Parabola is called DIAMETER. Equation to the diameter of a parabola is $y = 2a/m$, where m = slope of parallel chords.

S. IMPORTANT HIGHLIGHTS

(a) If the tangent & normal at any point 'P' of the parabola intersect the axis at T & G then ST = SG = SP where 'S' is the focus. In other words the tangent and the normal at a point P on the parabola are the bisectors of the angle between the focal radius SP & the perpendicular from P on the directrix. From this we conclude that all rays emanating from S will become parallel to the axis of the parabola after reflection.



(b) The portion of a tangent to a parabola cut off between the directrix & the curve subtends a right angle a the **focus**.

(c) The tangents at the extremities of a focal chord intersect at right angles on the directrix, and a circle on any focal chord as diameter touches the directrix. Also a circle on any focal radii of a point P $(at^2, 2at)$ as diameter touches the tangent at the vertex and intercepts a chord of length $(a\sqrt{1+t^2})$ on a normal at the point P.

(d) Any tangent to a parabola & the perpendicular on it from the focus meet on the tangent at the vertex.

(e) Semi latus rectum of the parabola $y^2 = 4ax$, is the harmonic mean between segments of any focal chord of the parabola is : $2a = \frac{2bc}{b+c}$ i.e. $\frac{1}{b} + \frac{1}{c} = \frac{1}{a}$.

Ex.21 The common tangent of the parabola $y^2 = 8ax$ and the circle $x^2 + y^2 = 2a^2$ is

Sol. Any tangent to parabola is $y = mx + \frac{2a}{m}$

Solving with the circle $x^2 + (mx + \frac{2a}{m})^2 = 2a^2 \Rightarrow x^2(1 + m^2) + 4ax + \frac{4a^2}{m^2} - 2a^2 = 0$

$B^2 - 4AC = 0$ gives $m = \pm 1$ Tangent $y = \pm x \pm 2a$

Ex.22 If the tangent to the parabola $y^2 = 4ax$ meets the axis in T and tangent at the vertex A in Y and the rectangle TAYG is completed, show that the locus of G is $y^2 + ax = 0$.

Sol. Let $P(at^2, 2at)$ be any point on the parabola $y^2 = 4ax$.

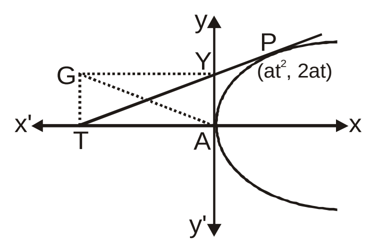
Then tangent at $P(at^2, 2at)$ is $ty = x + at^2$

Since tangent meet the axis of parabola in T and tangent at the vertex in Y.

$\therefore$ Co-ordinates of T and Y are $(-at^2, 0)$ and $(0, at)$ respectively.

Let co-ordinates of G be (x_1, y_1). Since TAYG is rectangle.

$\therefore$ Mid-points of diagonals TY and GA is same



$\Rightarrow \frac{x_1 + 0}{2} = \frac{-at^2 + 0}{2} \Rightarrow x_1 = -at^2$ ....(i) and $\frac{y_1 + 0}{2} = \frac{0 + at}{2} \Rightarrow y_1 = at$ ....(ii)

Eliminating t from (i) and (ii) then we get $x_1 = -a\left(\frac{y_1}{a}\right)^2$ or $y_1^2 = -ax_1$ or $y_1^2 + ax_1 = 0$

$\therefore$ The locus of $G(x_1, y_1)$ is $y^2 + ax = 0$

Ex.23 If $P(-3, 2)$ is one end of the focal chord PQ of the parabola $y^2 + 4x + 4y = 0$, then the slope of the normal at Q is

Sol. The equation of the tangent at $(-3, 2)$ to the parabola $y^2 + 4x + 4y = 0$ is

$2y + 2(x - 3) + 2(y + 2) = 0$ or $2x + 4y - 2 = 0 \Rightarrow x + 2y - 1 = 0$

Since the tangent at one end of the focal chord is parallel to the normal at the other end, the slope of the normal at the other end of the focal chord is $-\frac{1}{2}$.

Ex.24 Prove that the two parabolas $y^2 = 4ax$ and $y^2 = 4c(x - b)$ cannot have common normal, other than the axis unless $b/(a - c) > 2$.

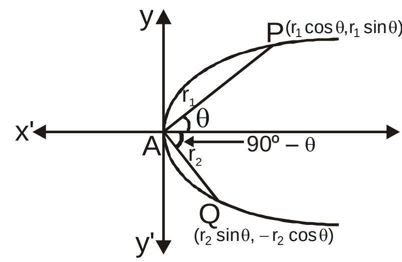
Sol. Given parabolas $y^2 = 4ax$ and $y^2 = 4c(x - b)$ have common normals. Then equation of normals in terms of slopes are $y = mx - 2am - am^3$ and $y = m(x - b) - 2cm - cm^3$ respectively then normals must be identical, compare the co-efficients

$1 = \frac{2am + am^3}{mb + 2cm + cm^3} \Rightarrow m[(c - a)m^2 + (b + 2c - 2a)] = 0, m \neq 0$ ($\because$ other than axis) and $m^2 = \dfrac{2a-2c-b}{c-a}$, $m = \pm\sqrt{\dfrac{2(a-c)-b}{c-a}}$

or $m = \pm\sqrt{\left(-2-\dfrac{b}{c-a}\right)}$ $\therefore$ $-2-\dfrac{b}{c-a} > 0$ or $-2+\dfrac{b}{a-c} > 0 \Rightarrow \dfrac{b}{a-c} > 2$

Ex.25 If r_1, r_2 be the length of the perpendicular chords of the parabola $y^2 = 4ax$ drawn through the vertex, then show that $(r_1r_2)^{4/3} = 16a^2\left(r_1^{2/3}+r_2^{2/3}\right)$.

Sol. Since chord are perpendicular, therefore if one makes an angle θ then the other will make an angle $(90^\circ - \theta)$ with x-axis



Let $AP = r_1$ and $AQ = r_2$

If $\angle PAX = \theta$ then $\angle QAX = 90^\circ - \theta$

$\therefore$ Co-ordinates of P and Q are $(r_1\cos\theta, r_1\sin\theta)$ and $(r_2\sin\theta, -r_2\cos\theta)$ respectively.

Since P and Q lies on $y^2 = 4ax$

$\therefore$ $r_1^2\sin^2\theta = 4ar_1\cos\theta$ and $r_2^2\cos^2\theta = 4ar_2\sin\theta \Rightarrow r_1 = \dfrac{4a\cos\theta}{\sin^2\theta}$ and $r_2 = \dfrac{4a\sin\theta}{\cos^2\theta}$

$$\therefore (r_1r_2)^{4/3} = \left(\frac{4a\cos\theta}{\sin^2\theta}.\frac{4a\sin\theta}{\cos^2\theta}\right)^{4/3} = \left(\frac{16a^2}{\sin\theta\cos\theta}\right)^{4/3} \quad \text{..........(i)}$$

and $16a^2.\left(r_1^{2/3}+r_2^{2/3}\right) = 16a^2\left\{\left(\dfrac{4a\cos\theta}{\sin^2\theta}\right)^{2/3}+\left(\dfrac{4a\sin\theta}{\cos^2\theta}\right)^{2/3}\right\}$

$$= 16a^2.(4a)^{2/3}\left\{\frac{(\cos\theta)^{2/3}}{(\sin\theta)^{4/3}}+\frac{(\sin\theta)^{2/3}}{(\cos\theta)^{4/3}}\right\} = 16a^2.(4a)^{2/3}\left\{\frac{\cos^2\theta+\sin^2\theta}{(\sin\theta)^{4/3}(\cos\theta)^{4/3}}\right\}$$

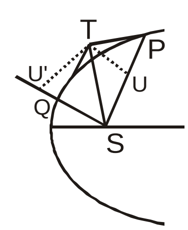
$$= \frac{16a^2.(4a)^{2/3}}{(\sin\theta\cos\theta)^{4/3}} = \left(\frac{16a^2}{\sin\theta\cos\theta}\right)^{4/3} = (r_1r_2)^{4/3} \quad \text{[from (i)]}$$

Ex.26 If the tangents at P and Q meet in T, prove that

(i) TP and TQ subtend equal angles at the focus S **(ii)** $ST^2 = SP.SQ$, and

(iii) the triangles SPT and STQ are similar.

Sol. Let P be the point $\left(at_1^2\ 2at_1\right)$, and Q be the point $\left(at_2^2\ 2at_2\right)$. Co-ordinates of T which is the point of intersection of tangents at P and Q is $\{at_1t_2,\ a(t_1+t_2)\}$



(i) The equation of SP is $y = \dfrac{2at_1}{at_1^2-1}(x-a)$ i.e. $\left(t_1^2-1\right)y - 2t_1x + 2at_1 = 0$

The perpendicular distance TU, from T on the straight line

$$= \left|\frac{a\left(t_1^2-1\right)(t_1+t_2)-2t.at_1t_2+2at_1}{\sqrt{\left(t_1^2-1\right)^2+4t_1^2}}\right| = \left|a\frac{\left(t_1^3-t_1^2t_2\right)+(t_1-t_2)}{t_1^2+1}\right| = |a(t_1-t_2)|.$$

Similarly TU has the same numerical value. The angles PST and QST are therefore equal.

(ii) We have SP = $a(1+t_1^2)$ and SQ = $a(1+t_2^2)$.

Also $ST^2 = (at_1t_2 - a)^2 + a^2 (t_1 + t_2)^2 = a^2[t_1^2t_2^2 + t_1^2 + t_2^2 + 1] = a^2(1+t_1^2)(1+t_2^2)$

Hence ST^2 = SP . SQ.

(iii) Since $\frac{ST}{SP} = \frac{SQ}{ST}$ and the angles TSP and TSQ are equal, the triangles SPT and STQ are similar, so that $\angle SQT = \angle STP$ and $\angle STQ = \angle SPT$.

Ex.27 The area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.

Sol. Let the three points on the parabola be $(at_1^2, 2at_1)$, $(at_2^2\ 2at_2)$ and $(at_3^2\ 2at_3)$

The area of the triangle formed by these points

$$\Delta_1 = \frac{1}{2}[at_1^2(2at_2 - 2at_3) + at_2^2(2at_3 - 2at_1) + at_2^2(2at_1 - 2at_2)] = -a^2(t_2 - t_3)(t_3 - t_1)(t_1 - t_2).$$

The points of intersection of the tangents at these points are

$(at_2t_3, a(t_2 + t_3))$, $(at_3t_1, a(t_3 + t_1))$ and $(at_1t_2, a(t_1 + t_2))$

The area of the triangle formed by these three points

$$\Delta_1 = \frac{1}{2}\{at_2t_3(at_3 - at_2\} + at_3t_1(at_1 - at_3) + at_1t_2(at_2 - at_1)\} = \frac{1}{2}a^2(t_2 - t_3)(t_3 - t_1)(t_1 - t_2)$$

Hence $\Delta_1 = 2\Delta_2$

Ex.28 Prove that the orthocentre of any triangle formed by three tangents to a parabola lies on the directrix.

Sol. Let the equations to the three tangents be $t_1y = x + at_1^2$ .....(i)

$t_2y = x + at_2^2$ ....(ii) and $t_3y = x + at_3^2$ ......(iii)

The point of intersection of (ii) and (iii) is found, by solving them, to be $(at_2t_3, a(t_2 + t_3))$

The equation to the straight line through this point perpendicular to (i) is

$y - a(t_2 + t_3) = -t_1(x - at_2t_3)$ i.e. $y + t_1x = a(t_2 + t_3 + t_1t_2t_3)$ ....(iv)

Similarly, the equation to the straight line through the line intersection of (iii) and (i) perpendicular to (ii) is $y + t_2x = a(t_3 + t_1 + t_1t_2t_3)$ ....(v)

and the equation to the straight line through the intersection of (i) and (ii) perpendicular to (iii) is

$y + t_1x = a(t_1 + t_2 + t_1t_2t_3)$ ....(vi)

The point which is common to the straight lines (iv), (v) and (vi)

i.e. the orthocentre of the triangle, is easily seen to be the point whose coordinates are

$x = -a$, $y = a(t_1 + t_2 + t_3 + t_1t_2t_3)$ and this point lies on the directrix.